Math 261B Thurs. 9/10

$\mathbb{G}_a$ <u>unipotent</u> – only fin. dim. irr. rep is trivial K,
$\mathbb{G}_a \curvearrowright A$ is unipotent, any $\mathbb{G}_a \curvearrowright V$ is unipotent

$\mathbb{G}_m$ <u>reductive</u> – A is a $\oplus$ of irr. reps. $\iff$ $K\cdot 1 \subseteq A$ is a $\oplus$ summand
$\Rightarrow$ Every $\mathbb{G}_m \curvearrowright V$ is a $\oplus$ of irreducibles.

(G Linear)
Reductive + Unipotent $\Rightarrow$ G trivial.

<u>Thm.</u> Every linear algebraic group G has G/R_u reductive.

$$\mathcal{O}(G/R_u) = \mathcal{O}(G)^{R_u}$$

$$G \to G/R_u$$
$$\mathcal{O}(G) \leftarrow \mathcal{O}(G/R_u)$$

$G \curvearrowright V$ $\quad$ V^{R_u} is a G-submodule

$$0 \to W \to V \to V/W \to 0$$

?

$$0 \to \underset{\text{triv.}}{K} \to V \to \underset{\text{triv.}}{K} \to 0$$

obstacle is non-trivial extensions of R_u modules

Example $G = B < GL_2$ $\begin{pmatrix} t_1 & x \\ 0 & t_2 \end{pmatrix}$ Defining rep $V = K^2$

Ke_1 is a submodule $ge_1 = t_1 e_1$ $g \mapsto (t_1)$ K_{t_1}

$$0 \to K_{t_1} \to V \to K_{t_2} \to 0$$

$U = R_u = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ $B/U = T \cong \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$ $B \to T$

$B = T \ltimes U$

K_{t_1} isn't a $\oplus$ summand "because"

$0 \to K \to V \to K \to 0$ is a non-trivial extension of R_u modules.

The Lie algebra version is $\mathfrak{g}/\mathfrak{r}$ is semisimple $\Rightarrow$ finite dim'l reps are completely reducible.

What do reductive G look like?

Answer is same over all alg. closed K for alg.

also for $\mathbb{C}$ reductive Lie groups $\leftrightarrow$ are alg.

also for compact Lie $/\mathbb{R}$.

Algebraic tori $T \cong (\mathbb{G}_m)^n$ $\quad (t_1, \dots, t_n)$ $\qquad (\mathbb{C}^\times)^n$

$\cup$

$A = k[t_1^{\pm 1}, \dots, t_1^{\pm n}]$ $\qquad (U_1)^n \leftarrow$ torus

$\Delta t_i = t_i \otimes t_i$ $\qquad (t_1, \dots, t_n) \cdot (s_1, \dots, s_n) = (t_1 s_1, \dots, t_n s_n)$

t_i is grouplike $\qquad \underline{a} \cdot f = f(- \cdot a)$

$(a_1, \dots, a_n)$ acts on A by $t_i \mapsto t_i a_i$ $\qquad (t_1, \dots, t_n) \cdot (a_1, \dots, a_n)$

$K \cdot t_1^{\lambda_1} \cdots t_n^{\lambda_n}$ $\qquad \lambda \in \mathbb{Z}^n$

is an invariant subspace

with $\underline{a} \mapsto (a^\lambda)$ $\qquad A$ is the $\bigoplus_\lambda K \cdot t^\lambda$

$\Rightarrow$ T is reductive, all its irreps are 1-dimensional: $\underline{a} \mapsto a^\lambda$, $\lambda \in \mathbb{Z}^n$.

$T \to GL_1 = \mathbb{G}_m \quad \rightsquigarrow$ 1-dimensional characters

$G \underset{\psi}{\overset{\varphi}{\rightrightarrows}} \mathbb{G}_m$ $\qquad \varphi \cdot \psi$, φ^{-1} is again a 1-dimensional characters

$\Downarrow$

tensor reps. $\qquad K_\lambda \otimes K_\mu = K_{\lambda+\mu}$

Lattice of characters $X(T) \cong \mathbb{Z}^n$, $\quad A = \bigoplus$ of them all

$A = K \cdot X \qquad \lambda \mapsto t^\lambda, \quad \lambda + \mu \mapsto t^\lambda t^\mu$

$$\Delta t^\lambda = \prod \Delta(t_i^{\lambda_i}) = \prod_i (t_i^{\lambda_i} \otimes t_i^{\lambda_i}) = t^\lambda \otimes t^\lambda$$

1-dimensional characters of G = grouplike elements of $A = \mathcal{O}(G)$.

$$T \xrightarrow{\varphi} T' \qquad X(T') \xrightarrow{\psi} X(T) \qquad \dashrightarrow \qquad K \cdot X(T') \to K(X(T))$$

$$\lambda \mapsto \lambda \circ \varphi \qquad\qquad t^\lambda \mapsto t^{\psi(\lambda)}$$

$$T \overset{\varphi}{\dashrightarrow} T' \to \mathbb{G}_m \qquad\qquad \mathcal{O}(T') \xrightarrow[\text{hom}]{\text{Hopf alg}} \mathcal{O}(T)$$

$$T' \leftarrow T \quad \text{alg. gp hom.}$$

$(\text{alg. tori}/K)^{op} \cong$ lattices = f.g. free abelian groups

$$T \mapsto X(T)$$

$$\mathrm{Spec}(K \cdot X) \leftarrow X$$

$X = \mathrm{Hom}_{alg/K}(T, \mathbb{G}_m)$ characters

$$X(T)^* = \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{Z}) \cong \mathrm{Hom}_{alg/K}(\mathbb{G}_m, T)$$

$\mathbb{Z} = X(\mathbb{G}_m)$

cocharacters (1-parameter "subgroups")

$X(T)^*$ paired with $X(T)$

$\beta \qquad\qquad \lambda$

$\langle \beta, \lambda \rangle \in \mathbb{Z}$

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, "\beta"] \arrow[rr, bend right, "\lambda\circ\beta"'] & T \arrow[r, "\lambda"] & \mathbb{G}_m \end{tikzcd}$$

$\lambda \circ \beta : t \mapsto t^r \qquad r \in \mathbb{Z}$

$r = \langle \beta, \lambda \rangle.$

$$\begin{tikzcd} T \arrow[r, "\varphi"] & T' \\ \mathbb{G}_m \arrow[u, "\beta"] \arrow[ur, dotted, "\varphi\circ\beta"'] & \end{tikzcd}$$

First ingredient of Cartan data for connected reductive G is a maximal alg. torus

$G \supset B \supset T \qquad\qquad B = T \ltimes U$

B: maximal solvable (Borel subgroup) $\qquad T$: max torus $\qquad U = R_u(B)$

$T \to X, X^*$ character + cochar lattices.

<u>Ex:</u> $G = GL_n$ $\qquad$ B = upper triangular matrices

U = upper uni-triangular

T = diagonal matrices

$$1 \to U \to B \underset{\hookleftarrow}{\to} T \qquad T = B/U \qquad B = T \ltimes U \qquad U = R_u(B)$$

$$\begin{pmatrix} t_1 & & x \\ & \ddots & \\ 0 & & t_n \end{pmatrix} \mapsto \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}$$

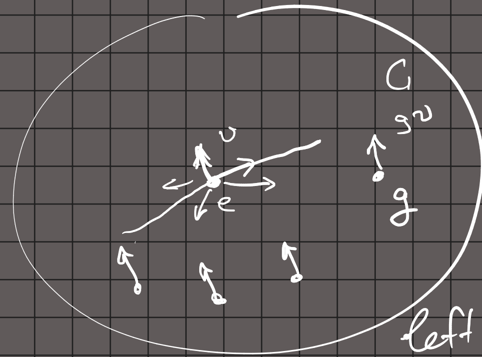
$X = \mathbb{Z}^n$ $\qquad t^\lambda$

$X^* \cong \mathbb{Z}^n$

$\mathfrak{g} = \mathfrak{gl}_n$ is tangent space to G at e

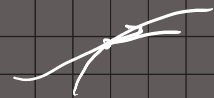
$\cong$

left invariant vector fields



a, b vector fields

ab, ba just 2^{nd} order differential operators $\qquad g = \xi(f)$



left-invariant vector field ξ

$[a, b] = ab - ba$ is a vector field

Gives $\mathfrak{g}$ a Lie algebra structure.

$G \hookrightarrow G \times G \circlearrowright G$

Also, G acts on itself: on right $g \cdot h = hg^{-1}$ on left $g \cdot h = gh$,

by conjugation $g \cdot h = ghg^{-1}$ $\leftarrow$ fixes e

${}^g h$

$\Rightarrow$ $G \curvearrowright G$ by conj. $\dashrightarrow$ $G \curvearrowright T_e G = \mathfrak{g}$ $\leftarrow$ algebraic action

Adjoint action Ad of G on $\mathfrak{g}$

$\mathrm{Ad}: G \curvearrowright \mathfrak{g}$ for GL_n, how $T \curvearrowright \mathfrak{g}$

$\mathfrak{g}^*$ is spanned by local coordinates at e.

$$\begin{pmatrix} 1+x_{11} & x_{12} & \\ x_{21} & 1+x_{22} & \\ & & \ddots \end{pmatrix} = I + \begin{pmatrix} x_{11} & x_{12} & \\ & \ddots & \\ & & \ddots \end{pmatrix}$$

e is $x_{ij} = 0$

$z = \det(I + \underline{X})^{-1}$

$= 1 +$ linear term + higher terms.

$$\frac{\partial}{\partial x_{ij}}\Big|_{x=0} = \partial x_{ij}$$ basis of tangent vectors

dual to local coordinates x_{ij}

$z \in \mathfrak{g}^*$

x_{ij} are linear coordinates of $\mathfrak{g}$

Ad G

$$g(I + x)g^{-1} = 1 + gxg^{-1} \qquad \mathfrak{g} = M_n$$

Ad is $\boxed{G \curvearrowright M_n \text{ by conjugation}}$ ✓

GL_n